

An iterative observer for boundary estimation for an infinite-dimensional elliptic Cauchy problem

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Abstract: We design and prove the convergence of an iterative observer for the boundary estimation problem for an elliptic equation namely Cauchy problem for Laplace equation. The Laplace equation is formulated as a first order system of equations in one of the space variables with state operator matrix \mathcal{A} that maps from $D(\mathcal{A})$ to X , where $D(\mathcal{A})$ and X are infinite-dimensional Sobolev spaces. The convergence of proposed iterative observer is established using semigroup theory and the concept of observability for infinite dimensional systems. We prove that \mathcal{A} generates a strongly continuous semigroup \mathbb{T} on a proper dense subspace of X . We also show that for a given particular boundary observation operator \mathcal{C} , pair $(\mathcal{C}, \mathcal{A})$ is final state and exact observable. Further a theorem is provided to prove a bi-implication between exact observability and existence of observer gain. Numerical simulations are provided toward the end to show efficiency of the algorithm.

1 Introduction

In this paper we consider the observer design for Cauchy problem for Laplace equation with Cauchy data prescribed on one of the boundaries using one of the space variables as a time-like variable. Let Ω be a rectangular domain in \mathbb{R}^2 with boundaries $\Gamma_T, \Gamma_B, \Gamma_L$ and Γ_R as shown in Figure 1 such that $\bar{\Omega} = \Omega \cup \Gamma_T \cup \Gamma_B \cup \Gamma_L \cup \Gamma_R$, $\Omega = (0, a) \times (0, b)$ and $\Gamma_T \cap \Gamma_B \cap \Gamma_L \cap \Gamma_R = \emptyset$. Cauchy problem for Laplace equation is defined as,

Find $u(x)$ on Γ_B :

$$\begin{cases} \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 & \text{in } \Omega, \\ u = f(x) & \text{on } \Gamma_T, \\ \frac{\partial u}{\partial n} = g(x) & \text{on } \Gamma_T, \end{cases} \quad (1)$$

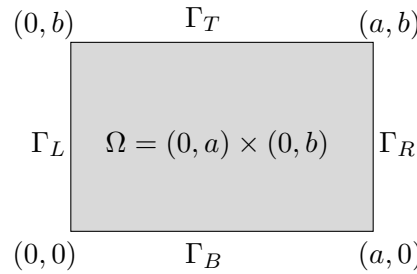


Figure 1: Rectangular domain Ω .

with homogeneous Dirichlet, Neumann or Robin type of side boundaries, f and g are given sufficiently smooth and $\frac{\partial}{\partial n}$ represents the normal derivative to the top boundary Γ_T . For consistent Cauchy data on Γ_B problem (1) has a nice analytical solution [1]. However, the objective here is to explore the possibility of developing an observer-like algorithm in space by representing Laplace equation as an infinite-dimensional linear state-space-like system. The goal of this work is to extend the dynamical theory concept of state-observer to steady state boundary value problems without introducing a notion of time and to explore the possibility of potential advantages over traditional methods of solution.

Cauchy problem for Laplace equation has been a fundamental problem of interest in many diverse areas of science and engineering. For example non-destructive testing applications in mechanics, where we are interested in finding inside cracks from boundary measurements [2]. Other applications include finding the actual heart potential from electrocardiogram (ECG) data collected on the body torso. Finding the actual heart potential is vital to understand the functionality of heart valves [3, 4]. Further there are some geophysical applications [5]. The mathematical problem has a history of more than hundred years and readers may refer to wide range of existing efficient numerical solution techniques for elliptic Cauchy problems, e.g. [6, 7, 8, 9, 10, 11, 12, 13].

The purpose of this study is to develop an iterative observer for Cauchy problem for Laplace equation by considering Laplace equation as a first order state equation in one of the space variables. State observer is a well-known concept in the estimation theory of linear dynamical systems [14, 15]. In this paper, unlike previous observer-based techniques for boundary estimation for Laplace equation system where an extra time variable was introduced [16, 17], a new method based on considering one of the spatial variables as a time-like variable is developed. This challenging new adaptation allows tackling the elliptic boundary value problem using linear estimation techniques by constructing a state observer without modifying the problem's core, i.e. the problem is still elliptic (in the steady state). Further the disadvantage of tackling elliptic problem as a parabolic one by introducing time variable is the computational cost of introducing a new time variable.

2 Notations and definitions

In this section, let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$. If X and Y are two Hilbert spaces then $\mathcal{L}(X, Y)$ denotes the space of linear operators from X to Y with induced norm. Further $\mathcal{L}(X) = \mathcal{L}(X, X)$. Let an infinite dimensional linear dynamical system be presented in state space representation as,

$$\dot{\xi}(x) = \mathcal{A}\xi(x); \quad y(x) = \mathcal{C}\xi(x); \quad (2)$$

such that " $\dot{\cdot}$ " represents partial derivative with respect to x , ξ be a state vector, $\mathcal{A} : D(\mathcal{A}) \rightarrow X$ be the state operator matrix, $\mathcal{C} \in \mathcal{L}(X, Y)$ be the observation operator with observation space Y .

Definition 1. [18] A family $\mathbb{T} = (\mathbb{T}_x)_{x \geq 0}$ of operators in $\mathcal{L}(X)$ defines a strongly continuous semigroup (C_0 -Semigroup) on X if,

1. $\mathbb{T}_0 = I$, (identity)
2. $\mathbb{T}_{x+w} = \mathbb{T}_x \mathbb{T}_w$, $\forall x, w \geq 0$, (semigroup property)
3. $\lim_{x \rightarrow 0^+} \|\mathbb{T}_x \xi - \xi\| = 0 \quad \forall \xi \in X$. (strong continuity)

Definition 2. Let $\mathcal{C} \in \mathcal{L}(X, Y)$ be the observation operator. For all $\bar{x} > 0$, let $\Psi_{\bar{x}} \in \mathcal{L}(X, L^2([0, \bar{x}]; Y))$ be the output map operator for the system (2) such that,

$$(\Psi_{\bar{x}} \xi(0))(x) = \begin{cases} \mathcal{C} \mathbb{T}_{\bar{x}} \xi(0) & \forall x \in [0, \bar{x}], \\ 0 & \forall x > \bar{x}. \end{cases} \quad (3)$$

Definition 3. [18] Let \mathbb{T} be the strongly continuous semigroup on space X with generator $\mathcal{A} : D(\mathcal{A}) \rightarrow X$ and $\mathcal{C} \in \mathcal{L}(X, Y)$ be the observation operator. The pair $(\mathcal{C}, \mathcal{A})$ is exactly observable in \bar{x} if $\Psi_{\bar{x}}$ is bounded from below.

Using the density of $D(\mathcal{A}^\infty)$ in X the definition of exact observability of the pair $(\mathcal{C}, \mathcal{A})$ is equivalent to the fact that there exists $k_{\bar{x}} > 0$ such that,

$$\int_0^{\bar{x}} \|\Psi_x \xi(0)\|^2 dx \geq k_{\bar{x}}^2 \|\xi(0)\|^2 \quad \forall \xi(0) \in X. \quad (4)$$

Definition 4. [18] Pair $(\mathcal{C}, \mathcal{A})$ as defined above is final state observable in \bar{x} if there exists a constant $k_{\bar{x}} > 0$ such that,

$$\|\Psi_{\bar{x}} \xi(0)\| \geq k_{\bar{x}} \|\mathbb{T}_{\bar{x}} \xi(0)\| \quad \forall \xi(0) \in X. \quad (5)$$

Remark 1. For $\bar{x} \rightarrow 0$, using the strong continuity of operator semigroup \mathbb{T} we can see that definitions in equation (4) and (5) converge.

Definition 5. [18] Pair $(\mathcal{C}, \mathcal{A})$ as defined above is approximately observable in \bar{x} if $\text{Ker} \Psi_{\bar{x}} = \{0\}$.

Further a remark from [18] suggests that exact observability implies the other two concepts of observability.

Definition 6. Let $x_1, x_2, x_3 \in [c, d]$ for all $c, d \in \mathbb{R}$ and $d > c$, then

$$x_1 \underset{(d-c)}{\diamond} x_2 \underset{(d-c)}{\diamond} x_3 = x, \quad \forall x \in [c, d] \cup [c, d] \cup [c, d]. \quad (6)$$

Above concatenation, over the interval of length $d - c$, suggests various iterations over the interval $[c, d]$ and specifically x_3 represents $x \in [c, d]$ over the third iteration. In general, let us define x_m as $x \in [c, d]$ for m -th iteration over the interval $[c, d]$ for $m \geq 1$.

Now let $s \in [0, \pi/4]$ without loss of generality, $\mathcal{A} : D(\mathcal{A}) \rightarrow X$ be an unbounded differential operator matrix given as,

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -\frac{\partial^2}{\partial s^2} & 0 \end{pmatrix}, \quad (7)$$

such that,

$$X = H_{\Gamma_T}^1 \left(0, \frac{\pi}{4}\right) \times L^2 \left(0, \frac{\pi}{4}\right), \quad (8)$$

$$D(\mathcal{A}) = \left[H^2 \left(0, \frac{\pi}{4}\right) \cap H_{\Gamma_T}^1 \left(0, \frac{\pi}{4}\right) \right] \times H_{\Gamma_T}^1 \left(0, \frac{\pi}{4}\right), \quad (9)$$

where,

$$H_{\Gamma_T}^1 \left(0, \frac{\pi}{4}\right) = \left\{ f \in H^1 \left(0, \frac{\pi}{4}\right) \mid \frac{df}{ds}(0) = 0 \right\}, \quad (10)$$

here X is a Hilbert space with scalar product given by,

$$\left\langle \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\rangle = \int_0^{\frac{\pi}{4}} \frac{dq_1}{ds}(s) \frac{d\bar{p}_1}{ds}(s) ds + \int_0^{\frac{\pi}{4}} q_1(s) \bar{p}_1(s) ds + \int_0^{\frac{\pi}{4}} q_2(s) \bar{p}_2(s) ds. \quad (11)$$

It can be seen that $D(\mathcal{A}^\infty)$ is dense in X .

Let us assume that for all $n \in \mathbb{Z}$, $\Phi_n(s)$ be an infinite set of smooth orthonormal basis in X (existence of such a basis is provided in Theorem 1. below). Let $N \in \mathbb{Z}$, let us define another space V as follows,

$$V = C_N^\infty \left(0, \frac{\pi}{4}\right) = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in X : \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \sum_{n \in \mathbb{Z}} \begin{pmatrix} (f_1)_n \\ (f_2)_n \end{pmatrix}, \right. \\ \left. \left\langle \Phi_n, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle = 0 \quad \forall N < n < \infty, \right\}. \quad (12)$$

It can be seen that $V \subset C^\infty \subset X$. V also forms a Hilbert space with respect to the inner product defined by equation (11). Further V is a dense proper subspace of C^∞ and also of X with respect to this norm. The density argument is easy to verify by assuming a small $\epsilon > 0$ and a vector $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in X$ and then taking a sequence $\sum_{n \in \mathbb{Z}} \begin{pmatrix} (f_1)_n \\ (f_2)_n \end{pmatrix} \in V$ such that $\left\| \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \sum_{n \in \mathbb{Z}} \begin{pmatrix} (f_1)_n \\ (f_2)_n \end{pmatrix} \right\| < \epsilon$ for some $N < \infty$ and $\|\cdot\|$ defined by equation (11), along with the fact that C^∞ is dense in L^p .

3 Observer design

We propose to write down the Laplace equation in rectangular coordinates as given in (1) as a first order state equation by introducing two new auxiliary variables ξ_1, ξ_2 as follows,

$$\begin{cases} \xi_1(x, y) = u(x, y), \\ \xi_2(x, y) = \frac{\partial u}{\partial x}, \end{cases} \quad (13)$$

and the resulting equation can now be written as,

$$\frac{\partial \xi}{\partial x} = \mathcal{A}\xi, \quad (14)$$

where,

$$\xi = \begin{pmatrix} \xi_1(x, y) \\ \xi_2(x, y) \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & 1 \\ -\frac{\partial^2}{\partial y^2} & 0 \end{pmatrix}. \quad (15)$$

ξ_1 and ξ_2 are called state variables and using these new variables, problem (1) can be written in equivalent form as,

Find $\xi_1(x, y)$ on Γ_B :

$$\begin{cases} \frac{\partial \xi}{\partial x} = \mathcal{A}\xi & \text{in } \Omega, \\ \mathcal{C}\xi(x) = \xi_1(x) = f(x) & \text{on } \Gamma_T, \\ \frac{\partial \xi_1}{\partial y} = g(x) & \text{on } \Gamma_T, \\ \xi_1 = 0 \text{ and/or } \frac{\partial \xi_1}{\partial x} = 0 & \text{on } \Gamma_{L \cup R}. \end{cases} \quad (16)$$

Boundary value problem in this form has a first order state equation in variable x and overdetermined data is available on Γ_T . Before the introduction of iterative observer equations, let us assume that left hand boundary Γ_B is connected to right hand boundary Γ_R to have the notion of infinite time-like variable x over the rectangular domain. The reason for having such an assumption is that we are trying to develop an observer using space as time and hoping that this observer will converge asymptotically in time-like variable x . Let m be a non-negative integer index of iteration over the domain Ω in x -direction. Let x_m , as given in Definition (6), represents $x \in [0, a]$ for the m -th iteration over the interval $[0, a]$. After introducing iteration index m , now an observer can be developed as follows,

Main result

Theorem 1. For consistent Cauchy data, boundary value problem given in (17) asymptotically ($m \rightarrow \infty$) converges to the true solution of boundary value problem (16).

$$\begin{cases} \frac{\partial}{\partial x} \hat{\xi}(x_m, y) = \mathcal{A} \hat{\xi}(x_m, y) - \mathcal{K} \mathcal{C}(\hat{\xi}(x_m, y) - \xi) & \text{in } \Omega, \\ \frac{\partial}{\partial y} \hat{\xi}_1(x_m, y) = g(x) & \text{on } \Gamma_T, \\ \frac{\partial^2}{\partial y^2} \hat{\xi}_1(x_m, y) = -\frac{\partial^2}{\partial x^2} \hat{\xi}_1(x_m, y) - \mathcal{K} \mathcal{C}(\hat{\xi}(x_m, y) - \xi) & \text{on } \Gamma_B, \\ \hat{\xi}(x_m, y) |_{\text{initial}} = \hat{\xi}(x_{m-1}, y) & \text{in } \bar{\Omega}, \end{cases} \quad (17)$$

where " $\hat{\cdot}$ " represents estimated quantity and $\hat{\xi}(x_m, y) |_{\text{initial}}$ represents the estimate over the whole domain $\bar{\Omega}$ at the start of m -th iteration. Observer starts at index $m = 1$ which represents first iteration. $\hat{\xi}(x_0, y)$ is initial guess at the start of the first iteration over the whole domain $\bar{\Omega}$. Any value of initial guess $\hat{\xi}(x_0, y)$ can be chosen at the start of first iteration. For each subsequent iteration, result of the previous iteration is used as initial estimate as given in the last equation in (17). Third equation in (17) is the assumption that Laplace equation is valid on the bottom boundary and this provides necessary boundary condition required on Γ_B . \mathcal{C} is the observation operator such that $\mathcal{C}\xi = \xi_1|_{\Gamma_T}$. \mathcal{K} is the correction operator chosen in such a way that state estimation error on Γ_T given by $(\mathcal{C}\hat{\xi}(x_m, y) - \mathcal{C}\xi)$ converges to zero exponentially.

3.1 Preliminary analysis

Before moving to the formal proof of theorem (1) we note that the solution of first order equation in system (16) leads to the concept of semigroup generated by unbounded differential operator matrix \mathcal{A} . We study the exponential of \mathcal{A} using the functional analysis framework from section 2. Further using the concept of observability for infinite dimensional systems, we establish that pair $(\mathcal{C}, \mathcal{A})$ is final state and exact observable.

Semigroup generated by \mathcal{A}

Theorem 2. For $\mathcal{A} : D(\mathcal{A}) \rightarrow X$ (as given in (8), (9) and (15)) there exists an infinite set of orthonormal eigenvectors and corresponding eigenvalues. Furthermore \mathcal{A} generates a strongly continuous semigroup for vectors $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in V$ (V as given in (12)).

Proof. Let $n \in \mathbb{Z}$ set of all integers such that,

$$\phi_n(y) = \begin{pmatrix} \alpha_n \varphi_n(y) \\ \beta_n \varphi_n(y) \end{pmatrix}, \quad (18)$$

be the eigenvectors of operator \mathcal{A} and λ_n be the eigenvalues such that,

$$\begin{aligned} \mathcal{A} \begin{pmatrix} \alpha_n \varphi_n \\ \beta_n \varphi_n \end{pmatrix} &= \lambda_n \begin{pmatrix} \alpha_n \varphi_n \\ \beta_n \varphi_n \end{pmatrix}, \\ \begin{pmatrix} \beta_n \varphi_n \\ -\frac{\partial^2}{\partial y^2} (\alpha_n \varphi_n) \end{pmatrix} &= \lambda_n \begin{pmatrix} \alpha_n \varphi_n \\ \beta_n \varphi_n \end{pmatrix}. \end{aligned} \quad (19)$$

Assuming that α_n, β_n do not depend on y , second equation above suggests that we are interested in finding the eigenfunctions of Laplacian operator $-\frac{\partial^2}{\partial y^2}$. This signifies that unknown eigenfunctions $\varphi_n \in C^\infty$. Solving two equations in (19) gives,

$$\lambda_n = \frac{\beta_n}{\alpha_n}, \quad (20)$$

$$\varphi_n(y) = C_1 \cos(\lambda_n y), \quad (21)$$

where α_n and β_n depend on n . C_1 and λ_n are chosen such that $\varphi_n(y)$ in (21) forms an orthonormal basis in $L^2(0, \frac{\pi}{4})$, with $C_1 = -\sqrt{\frac{8}{\pi}}$, $\alpha_n = 1$ and $\beta_n = \lambda_n = 6 - 8n$. Finally an orthonormal set of eigenvectors can be formed in X with respect to norm defined by (11) as,

$$\Phi_n(y) = \rho_n \phi_n(y) = \rho_n \begin{pmatrix} \alpha_n \varphi_n(y) \\ \beta_n \varphi_n(y) \end{pmatrix}, \quad (22)$$

where $\rho_n = \frac{1}{\sqrt{2}\beta_n}$ is a normalization factor. Now let us try to write semigroup generated by operator matrix \mathcal{A} can be written as an infinite series,

$$\sum_{n \in \mathbb{Z}} e^{\lambda_n x} \left\langle \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix}, \Phi_n(y) \right\rangle \Phi_n(y), \quad \forall \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix} \in X. \quad (23)$$

For $x = 0$ the above infinite series is clearly convergent, whereas for $x \rightarrow 0^+$ the limit does not exist. Further we note that above series expression (23) satisfies identity and semigroup properties as given in Definition 1, however it lacks strong continuity, except if we assume that the projection terms in angle brackets above are non-zero for some large $n \in \mathbb{Z} : n < \infty$. Now with the introduction of this assumption the limit $x \rightarrow 0^+$ exists. This also reveals a historical fact about solving Cauchy problems for steady state heat equation that unique and stable solutions does not exist for non-smooth data [1]. Thus with this additional smoothness assumption that $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in V$ equation (24) represents the strongly continuous semigroup generated by operator matrix \mathcal{A} .

$$\mathbb{T}_x \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix} = \sum_{n \in \mathbb{Z}} e^{\lambda_n x} \left\langle \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix}, \Phi_n(y) \right\rangle \Phi_n(y), \quad \forall \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix} \in V. \quad (24)$$

This implies,

$$\begin{aligned} \mathbb{T}_x \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix} &= \sum_{n \in \mathbb{Z}} e^{\lambda_n x} \rho_n \left(\alpha_n \left\langle \frac{dp_1}{dy}(y), \frac{d\varphi_n}{dy}(y) \right\rangle_{L^2\left(0, \frac{\pi}{4}\right)} + \alpha_n \langle p_1(y), \varphi_n(y) \rangle_{L^2\left(0, \frac{\pi}{4}\right)} \right. \\ &\quad \left. + \beta_n \langle p_2(y), \varphi_n(y) \rangle_{L^2\left(0, \frac{\pi}{4}\right)} \right) \Phi_n(y), \quad \forall \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix} \in V. \end{aligned} \quad (25)$$

□

System observability

Proposition 1. Let \mathbb{T} be the strongly continuous semigroup generated by operator matrix \mathcal{A} under some assumptions as given in theorem (2). There exists arbitrarily small $\epsilon > 0$ such that if $\|\bar{x} - x\| < \epsilon$ then pair $(\mathcal{C}, \mathcal{A})$ is final state observable and hence, using remark (1), exactly observable in $\bar{x} > 0$ at a particular x , where $\mathcal{C} \in \mathcal{L}(V, Y)$ and $Y = \mathbb{R}$.

Proof. Let $\xi(0)$ be the initial guess at $x = 0$, given by,

$$\xi(0) = \begin{pmatrix} \xi_1(0) \\ \xi_2(0) \end{pmatrix} = \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix}. \quad (26)$$

Then,

$$\mathcal{C}\mathbb{T}_x \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix} = \sum_{n \in \mathbb{Z}} e^{\lambda_n x} \left\langle \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix}, \Phi_n(y) \right\rangle \mathcal{C}\Phi_n(y), \quad \forall \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix} \in V. \quad (27)$$

This gives,

$$\left| \mathcal{C}\mathbb{T}_x \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix} \right|^2 = \left| \sum_{n \in \mathbb{Z}} e^{\lambda_n x} \left\langle \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix}, \Phi_n(y) \right\rangle \mathcal{C}\Phi_n(y) \right|^2.$$

Now $\mathcal{C}\Phi_n(y) = -\rho_n$. This gives,

$$\begin{aligned} \left| \mathcal{C}\mathbb{T}_x \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix} \right|^2 &= \left| \sum_{n \in \mathbb{Z}} -e^{\lambda_n x} \rho_n^2 \left\langle \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix}, \phi_n(y) \right\rangle \right|^2, \\ &\geq \left| -\sum_{n \in \mathbb{Z}} e^{\lambda_n x} \rho_n^2 \left\| \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix} \right\|^2 \|\phi_n(y)\|^2 \right|^2, \\ &\geq \sum_{n \in \mathbb{Z}} \left| e^{\lambda_n x} \rho_n^2 \|\phi_n(y)\|^2 \right|^2 \left\| \begin{pmatrix} p_1(y) \\ p_2(y) \end{pmatrix} \right\|^2, \end{aligned} \quad (28)$$

$\sum_{n \in \mathbb{Z}} \left| e^{\lambda_n x} \rho_n^2 \|\phi_n\|^2 \right|^2 > 0$ for all $x \geq 0$. Hence pair $(\mathcal{C}, \mathcal{A})$ is exactly observable for arbitrarily small \bar{x} . Further from definitions (3), (4) and remark (1), it is obvious that for $\epsilon \rightarrow 0$, $\bar{x} \rightarrow (x = 0)$ and exact observability implies final state observability and vice versa. □

3.2 Convergence analysis

After establishing the concept of strongly continuous semigroup generated by \mathcal{A} and the fact that pair $(\mathcal{C}, \mathcal{A})$ is final state and exact observable, we are all set to prove the main result.

Proof of the main result

Proof. Let us define state estimation error $\tilde{e}(x_m, y)$ as the difference of true state $\xi(x, y)$ from the one estimated $\hat{\xi}(x_m, y)$,

$$\tilde{e}(x_m, y) = \hat{\xi}(x_m, y) - \xi(x, y) = \begin{pmatrix} \tilde{e}_1(x_m, y) \\ \tilde{e}_2(x_m, y) \end{pmatrix} = \begin{pmatrix} \hat{\xi}_1(x_m, y) - \xi_1(x, y) \\ \hat{\xi}_2(x_m, y) - \xi_2(x, y) \end{pmatrix}. \quad (29)$$

Solution of the boundary value problem (16) with consistent boundary data provides $u = \xi_1$ over the whole domain $\bar{\Omega}$. Boundary value problem for the state estimation error can be given by subtracting problem (16) from the state observer equations (17) as follows,

For $m \geq 1$, find $\tilde{e}(x_m, y) = (\hat{\xi}(x_m, y) - \xi(x, y)) \in \bar{\Omega}$:

$$\begin{cases} \frac{\partial}{\partial x} \tilde{e}(x_m, y) = (\mathcal{A} - \mathcal{K}\mathcal{C})\tilde{e}(x_m, y) & \text{in } \Omega, \\ \frac{\partial}{\partial y} \tilde{e}_1(x_m, y) = 0 & \text{on } \Gamma_T, \\ \frac{\partial^2}{\partial y^2} (\hat{\xi}_1(x_m) - h(x)) = -\frac{\partial^2}{\partial x^2} (\hat{\xi}_1(x_m) - h(x)) - \mathcal{K}\mathcal{C}\tilde{e}(x_m) & \text{on } \Gamma_B, \\ \tilde{e}(x_m, y) |_{\text{initial}} = \tilde{e}(x_{m-1}, y) & \text{in } \bar{\Omega}. \end{cases} \quad (30)$$

Here $h(x)$ is the true analytical solution on Γ_B using consistent Cauchy data. Further using the assumption that Laplace equation is valid on Γ_B , the above system of error dynamic equation can also be written in an equivalent form as,

For $m \geq 1$, find $\tilde{e}(x_m, y) \in \bar{\Omega}$:

$$\begin{cases} \frac{\partial}{\partial x} \tilde{e}(x_m, y) = (\mathcal{A} - \mathcal{K}\mathcal{C})\tilde{e}(x_m, y) & \text{in } \bar{\Omega} \setminus \Gamma_T, \\ \frac{\partial}{\partial y} \tilde{e}(x_m, y) = 0 & \text{on } \Gamma_T, \\ \tilde{e}(x_m, y) |_{\text{initial}} = \tilde{e}(x_{m-1}, y) & \text{in } \bar{\Omega}. \end{cases} \quad (31)$$

First equation in (31) is an ODE in variable x and solution to this ODE problem is a decaying exponential as follows,

$$\tilde{e}(x_m, \cdot) = e^{(\mathcal{A} - \mathcal{K}\mathcal{C})x_m} \tilde{e}(x_0, \cdot) \quad m \geq 1, \quad (32)$$

for a particular iteration index m , x_m is $x \in [0, a]$ over m -th iteration. Observer gain operator \mathcal{K} can be chosen in a way that $\mathcal{A} - \mathcal{K}\mathcal{C}$ is dissipative and state estimation error $\tilde{e}(x_m, \cdot)$, for a number of iterations over the whole domain, asymptotically converges to zero for any initial value of $\tilde{e}(x_0, \cdot)$. \square

3.3 Existence of observer gain \mathcal{K}

Theorem 3. Let \mathcal{A} as given in (15) be the generator a strongly continuous semigroup and $\mathcal{C} \in L(V, Y)$ an observation operator (V as given in (12) and $Y = \mathbb{R}$), then the following assertions are equivalent.

1. There exists a positive definite self-adjoint operator product $\mathcal{K}\mathcal{C} \in L(V)$ where $\mathcal{K} \in L(Y, V)$ such that $\mathcal{A} - \mathcal{K}\mathcal{C}$ generates a maximally dissipative semigroup.
2. There exists arbitrarily small $\epsilon > 0$ such that if $\|\bar{x} - x\| < \epsilon$ then pair $(\mathcal{C}, \mathcal{A})$ is exactly observable in \bar{x} .

Proof. Given self-adjoint positive definite operator product $\mathcal{K}\mathcal{C} \in L(V)$, let us denote by \mathbb{S} and \mathbb{T} the semigroups generated by $\mathcal{A} - \mathcal{K}\mathcal{C}$ and \mathcal{A} respectively.

1 \Rightarrow 2:

Assume \mathbb{S} is dissipative, let us show the observability inequality, that is, there exists $\bar{x}, k > 0$ such that,

$$\int_0^{\bar{x}} \|\mathcal{C}\mathbb{T}_x e_0\|^2 \geq k^2 \|e_0\|^2 \quad \forall e_0 \in V, \quad (33)$$

\mathcal{A} is densely defined so the above inequality is enough to prove exact observability. Given $e_0 \in D(\mathcal{A})$, $e(x) = \mathbb{S}_x e_0$ presents the unique solution of,

$$\begin{cases} \frac{\partial e}{\partial x} = (\mathcal{A} - \mathcal{K}\mathcal{C})e(x), \\ e(0) = e_0. \end{cases} \quad (34)$$

Multiplying first equation in (34) by $e(x)$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dx} \|e(x)\|^2 &= \operatorname{Re} \left\langle \frac{\partial e}{\partial x}, e(x) \right\rangle, \\ &= \operatorname{Re} \langle (\mathcal{A} - \mathcal{K}\mathcal{C})e(x), e(x) \rangle, \end{aligned}$$

integrating both sides from 0 to \bar{x} gives,

$$\begin{aligned} \frac{1}{2} \int_0^{\bar{x}} \frac{d}{dx} \|e(x)\|^2 dx &= - \int_0^{\bar{x}} \alpha^2 dx, \\ \|e_0\|^2 - \|e(\bar{x})\|^2 &= 2 \int_0^{\bar{x}} \alpha^2 dx. \end{aligned} \quad (35)$$

where $\operatorname{Re} \langle (\mathcal{A} - \mathcal{K}\mathcal{C})e(x), e(x) \rangle = -\alpha^2$, as $\mathcal{A} - \mathcal{K}\mathcal{C}$ generates a dissipative semigroup and $\alpha^2 > 0$. Let $e(x) = \gamma(x) + \zeta(x)$ such that $\|e\|^2 > \{\|\gamma\|^2; \|\zeta\|^2\}$ and $\langle \gamma, \zeta \rangle > 0$, where $\gamma = \mathbb{T}_x e_0$ is the solution of,

$$\begin{cases} \frac{\partial \gamma}{\partial x} = \mathcal{A}\gamma(x), \\ \gamma(0) = e_0, \end{cases} \quad (36)$$

and ζ is the solution of,

$$\begin{cases} \frac{\partial \zeta}{\partial x} = \mathcal{A}\zeta(x) - \mathcal{K}\mathcal{C}e(x), \\ \zeta(0) = 0. \end{cases} \quad (37)$$

Further we can write,

$$\begin{aligned}
\operatorname{Re} \langle (\mathcal{K}\mathcal{C} - \mathcal{A})e, e \rangle &\leq \operatorname{Re} \langle (\mathcal{K}\mathcal{C} - \mathcal{A})\gamma, \gamma \rangle, \\
&\leq \operatorname{Re} \langle \mathcal{K}\mathcal{C}\gamma, \gamma \rangle - \operatorname{Re} \langle \mathcal{A}\gamma, \gamma \rangle, \\
&\leq 2\operatorname{Re} \langle \mathcal{K}\mathcal{C}\gamma, \gamma \rangle, \\
\alpha^2 &\leq 2\operatorname{Re} \langle \mathcal{K}\mathcal{C}\gamma, \mathcal{K}\mathcal{C}\gamma \rangle,
\end{aligned}$$

Now using Cauchy-Schwarz inequality we have,

$$\operatorname{Re} \langle \mathcal{K}\mathcal{C}\gamma, \mathcal{K}\mathcal{C}\gamma \rangle \leq \|\mathcal{K}\mathcal{C}\gamma(x)\|_V^2. \quad (38)$$

This implies,

$$\alpha^2 \leq \|\mathcal{K}\mathcal{C}\gamma(x)\|_V^2. \quad (39)$$

From equation (35),

$$\|e_0\|^2 - \|e(\bar{x})\|^2 \leq 4 \int_0^{\bar{x}} \|\mathcal{K}\mathcal{C}\gamma(x)\|_V^2 dx. \quad (40)$$

Next, as \mathbb{S} is m-dissipative, we have,

$$\|e(\bar{x})\|^2 = \|\mathbb{S}e_0\|^2 \leq M^2 e^{-2\omega\bar{x}} \|e_0\|^2, \quad (41)$$

with $M^2 e^{-2\omega\bar{x}} < 1$. Finally,

$$k^2 \|e_0\|^2 \leq \int_0^{\bar{x}} \|\mathcal{C}\gamma(x)\|_Y^2 dx, \quad (42)$$

where, $k^2 = \frac{1 - M^2 e^{-2\omega\bar{x}}}{4 \|\mathcal{K}\|^2}$.

$2 \Rightarrow 1$:

We have $e(x) = \gamma(x) + \zeta(x)$, this implies,

$$\int_0^{\bar{x}} \langle (\mathcal{K}\mathcal{C} - \mathcal{A})\gamma(x), \gamma(x) \rangle dx \leq \int_0^{\bar{x}} \langle (\mathcal{K}\mathcal{C} - \mathcal{A})e(x), e(x) \rangle dx. \quad (43)$$

From equation (35),

$$\int_0^{\bar{x}} \langle \mathcal{K}\mathcal{C}\gamma(x), \gamma(x) \rangle dx \leq \frac{3}{2} \left(\|e_0\|^2 - \|e(\bar{x})\|^2 \right) + \int_0^{\bar{x}} \langle \mathcal{A}\gamma(x), \gamma(x) \rangle dx, \quad (44)$$

now from (36), we have,

$$\begin{aligned}
\operatorname{Re} \langle \mathcal{A}\gamma, \gamma \rangle &= \operatorname{Re} \left\langle \frac{\partial \gamma}{\partial x}, \gamma \right\rangle, \\
&= \frac{1}{2} \frac{d}{dx} \|\gamma(x)\|^2, \\
&= \frac{1}{2} \frac{d}{dx} \|e(x) - \zeta(x)\|^2, \\
&\leq \frac{1}{2} \frac{d}{dx} \|e(x)\|^2 + \frac{1}{2} \frac{d}{dx} \|\zeta(x)\|^2,
\end{aligned} \quad (45)$$

where,

$$\begin{aligned}
\frac{1}{2} \frac{d}{dx} \|\zeta(x)\|^2 &= \operatorname{Re} \left\langle \frac{\partial \zeta}{\partial x}, \zeta \right\rangle, \\
&= \langle \mathcal{A}\zeta - \mathcal{K}\mathcal{C}e, \zeta \rangle, \\
&= \langle (\mathcal{A} - \mathcal{K}\mathcal{C})\zeta, \zeta \rangle - \langle \mathcal{K}\mathcal{C}\gamma, \zeta \rangle, \\
&\leq \langle (\mathcal{A} - \mathcal{K}\mathcal{C})\zeta, \zeta \rangle, \\
&\leq \langle (\mathcal{A} - \mathcal{K}\mathcal{C})e, e \rangle, \\
&\leq \frac{1}{2} \frac{d}{dx} \|e(x)\|^2.
\end{aligned} \tag{46}$$

This implies,

$$\begin{aligned}
\int_0^{\bar{x}} \operatorname{Re} \langle \mathcal{A}\gamma, \gamma \rangle dx &\leq \int_0^{\bar{x}} \frac{d}{dx} \|e(x)\|^2 dx, \\
&\leq - \left(\|e_0\|^2 - \|e(\bar{x})\|^2 \right).
\end{aligned} \tag{47}$$

Now from equation (44) we have,

$$\begin{aligned}
\int_0^{\bar{x}} \langle \mathcal{K}\mathcal{C}\gamma(x), \gamma(x) \rangle dx &\leq \frac{1}{2} \left(\|e_0\|^2 - \|e(\bar{x})\|^2 \right), \\
\int_0^{\bar{x}} \langle \mathcal{C}\gamma(x), \mathcal{C}\gamma(x) \rangle dx &\leq \frac{1}{2\|\mathcal{K}\|} \left(\|e_0\|^2 - \|e(\bar{x})\|^2 \right), \\
\int_0^{\bar{x}} \|\mathcal{C}\gamma(x)\|_Y^2 dx &\leq \frac{1}{2\|\mathcal{K}\|} \left(\|e_0\|^2 - \|e(\bar{x})\|^2 \right),
\end{aligned} \tag{48}$$

above inequality is from the fact that $\langle \mathcal{K}\mathcal{C}\gamma, \gamma \rangle > 0$ and \mathcal{C} is a boundary observation operator. Now using observability inequality (42) we have,

$$\begin{aligned}
k^2 \|e_0\|^2 &\leq \frac{1}{2\|\mathcal{K}\|} \left(\|e_0\|^2 - \|e(\bar{x})\|^2 \right), \\
\|e(\bar{x})\|^2 &\leq \left(\frac{1}{2\|\mathcal{K}\|} - k^2 \right) \|e_0\|^2, \\
\|e(\bar{x})\|^2 &\leq \left(1 - \left(\frac{1 - M^2 e^{-2\omega\bar{x}}}{\|\mathcal{K}\|} \right) \right) \|e_0\|^2,
\end{aligned} \tag{49}$$

where $\left(1 - \left(\frac{1 - M^2 e^{-2\omega\bar{x}}}{\|\mathcal{K}\|} \right) \right) < 1$ because from observability inequality $M^2 e^{-2\omega\bar{x}} < 1$. Hence semigroup generated by $\mathcal{A} - \mathcal{K}\mathcal{C}$ is maximally dissipative. \square

In section 4, the observer is implemented numerically using fictitious points on the estimated solution boundary. Numerical results are presented in the last section.

4 Numerical implementation and results

First order state equation given in (16) can be discretized in variable θ using Forward Euler as follows,

$$\begin{aligned}\dot{\xi} &= \mathcal{A}\xi, \\ \frac{\xi^{n+1} - \xi^n}{\Delta x} &= \mathcal{A}\xi^n, \\ \xi^{n+1} &= (I + (\Delta x)\mathcal{A})\xi^n,\end{aligned}\tag{50}$$

here I is the identity matrix, n is the discrete index for variable x and Δx is the step size along x after discretization. Further equation (50) is discretized in variable y using second order accurate centered finite difference schemes to discretize the first and second order derivative terms. Cauchy data is available on the top boundary however for the bottom boundary Γ_B there's no data available and we assume that Laplace equation is valid on this boundary as given in equation (17). Numerically this condition can be implemented using fictitious points along the inner boundary as explained in the next section.

4.1 Boundary condition on Γ_B

As stated in the previous section, first order state equation can be thought of as an ODE with respect to variable x . Solution of this ODE is the state ξ over the whole vertical line, that is, $(y|_{\Gamma_B}, y|_{\Gamma_T})$. This can be thought of as 2D Laplace equation has been split into a series of 1D state equations. To solve this 1D state equation in variable x , initial condition over the whole interval $(y|_{\Gamma_B}, y|_{\Gamma_T})$ and boundary conditions on Γ_B and Γ_T are required. Any initial guess can be chosen as $(\mathcal{A} - \mathcal{K}\mathcal{C})$ will be dissipative and any initial guess dies out. Two boundary conditions on Γ_T are available, that is the Cauchy data on Γ_T . However on Γ_B it is assumed that Laplace equation is satisfied. That is,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \text{ on } \Gamma_B.\tag{51}$$

Equation (51) contains second order derivative in variable y . To discretize this second derivative using second order accurate centered finite difference discretization scheme on Γ_B , there needs to be a fictitious point [19] further outside the boundary Γ_B as shown in figure 2.

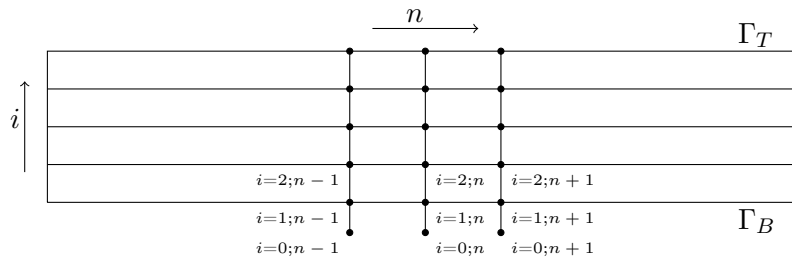


Figure 2: Domain Ω after discretization and fictitious points outside Γ_B , index $i = 0$ represents fictitious points.

Second equation in (50), after full discretization, can be written as,

$$\frac{(\xi_2)_i^{n+1} - (\xi_2)_i^n}{\Delta x} = -\frac{(\xi_1)_{i+1}^n - 2(\xi_1)_i^n + (\xi_1)_{i-1}^n}{(\Delta y)^2},\tag{52}$$

here i is the discrete index and (Δy) is the step size along variable y such that $i = 1$ on the bottom boundary Γ_B . Equation (52) on the bottom boundary Γ_B can be written as,

$$\frac{(\xi_2)_1^{n+1} - (\xi_2)_1^n}{\Delta x} = -\frac{(\xi_1)_2^n - 2(\xi_1)_1^n + (\xi_1)_0^n}{(\Delta y)^2}, \quad (53)$$

index $i = 0$ represents fictitious point and taking out this fictitious point gives,

$$(\xi_1)_0^n = 2(\xi_1)_1^n - (\xi_1)_2^n - \frac{(\Delta y)^2}{\Delta x} \{(\xi_2)_1^{n+1} - (\xi_2)_1^n\}. \quad (54)$$

$(\xi_1)_1^n, (\xi_1)_2^n, (\xi_2)_1^n$ and $(\xi_2)_1^{n+1}$ are given by the initial guess of the states over the whole domain. The algorithm is run for a number of iterations along x by using the solution of the previous iteration as a guess for the next until the final convergence is achieved. In the following subsection observer is presented in semi-discrete form and fictitious points method is used to tackle the boundary condition on Γ_{in} .

4.2 Observer in semi-discrete form

In the following, state observer presented in system of equations (17) is discretized only in variable x for simplicity.

$$\begin{cases} \hat{\xi}^{n+1,m} = (I + (\Delta x)\mathcal{A})\hat{\xi}^{n,m} - \mathcal{K}\mathcal{C}(\hat{\xi}^{n,m} - \xi^n) & \text{in } \Omega, \\ \frac{\partial}{\partial y}\hat{\xi}_1^{n,m} = g^n(x) & \text{on } \Gamma_T, \\ \frac{1}{2\Delta x}(\hat{\xi}_1^{n+1,m} - \hat{\xi}_1^{n-1,m}) = -\frac{\partial^2}{\partial y^2}\hat{\xi}_1^{n,m} - \mathcal{K}\mathcal{C}(\hat{\xi}^{n,m} - \xi^n) & \text{on } \Gamma_B, \\ \hat{\xi}^{n,m}|_{\text{initial}} = \hat{\xi}^{n,m-1} & \text{in } \bar{\Omega}, \end{cases} \quad (55)$$

again here $\hat{\cdot}$ represents estimated quantity and m is the index of iteration over the rectangular domain. $\hat{\xi}^{m,n}|_{\text{initial}}$ represents estimate for particular value of index n at the start of m^{th} iteration. Algorithm starts at $m = 1$ and index $m = 0$ represents the raw data over the whole mesh before start of the algorithm. At the start of the algorithm, any initial guess can be chosen over the whole domain and then solution of the previous iteration is used as a guess for subsequent iterations. Fictitious points are computed using formula given in (54) and are used in third equation in (55) to discretize the second order derivative with respect to y on Γ_B . Important point to note here is that true Neumann boundary condition is applied on the outer boundary and operators \mathcal{A}, \mathcal{K} and \mathcal{C} are continuous in variable y .

4.3 Algorithm step-by-step

- *Step 1:* Initialize mesh over the whole domain $\bar{\Omega}$ with $\hat{\xi}^{m=0} = \xi_0$.
- *Step 2:* For $m = 1$, start at a particular value of x ,
 - Compute the fictitious point value $(\xi_1)_0^{n,m=1}$ for particular value of n using equation (54).
 - Solve system of equations (55) to find estimate $\hat{\xi}^{n+1,m}$ over a particular vertical line.
 - Repeat the process of finding fictitious point from equation (54) and solving system of equations (55) for all n until one iteration on interval of length a on the rectangular domain shown in Figure. 1 is complete.

- *Step 3:* Repeat *Step 2* for $m \geq 2$ using result of $(m - 1)$ th iteration as a guess for m th iteration until convergence is achieved. That is, $\|\xi_1 - \hat{\xi}_1^m\|_{\Gamma_{out}} < \epsilon$.

4.4 State estimation error and computation of observer gain

State error boundary value problem in semi-discrete form can be written as,

For $m \geq 1$, find $e^{n,m} = (\hat{\xi}^{n,m} - \xi^n) \in \bar{\Omega}$:

$$\begin{cases} e^{n+1,m} = (\hat{\xi}^{n+1,m} - \xi^{n+1,m}) = (I + (\Delta x)\mathcal{A} - \mathcal{K}\mathcal{C})(\hat{\xi}^{n,m} - \xi^n) & \text{in } \bar{\Omega} \setminus \Gamma_T, \\ \frac{\partial}{\partial y} e_1^{n,m} = \frac{\partial}{\partial y} (\hat{\xi}_1^{n,m} - \xi_1^n) = 0 & \text{on } \Gamma_T, \\ e^{n,m}|_{initial} = e^{n,m-1} & \text{in } \bar{\Omega}. \end{cases} \quad (56)$$

Finally the state error difference equation after full discretization can be written as,

$$\mathbf{e}^{n+1,m} = (I + (\Delta x)A - KC) \mathbf{e}^{n,m} \quad \text{for } m \geq 1, \quad (57)$$

here A, K and C are discrete versions of operators \mathcal{A}, \mathcal{K} and \mathcal{C} respectively and \mathbf{e} is the state estimation error after full discretization. Given $(I + (\Delta x)A)$ and observation matrix C , gain matrix K can be computed using Ackermann's formula for pole placement in Matlab such that eigenvalues of $(I + (\Delta x)A - KC)$ are inside the unit circle on the complex plane [20].

4.5 Results and simulations

For all numerical and analytical solutions in this section, a rectangle domain $\Omega = (0, a) \times (0, b)$ with $a = 2\pi$ and $b = \frac{1}{2}$ is considered. To validate the observer approach a number of examples are presented as follows.

4.5.1 Example 1: Homogeneous Neumann side boundaries

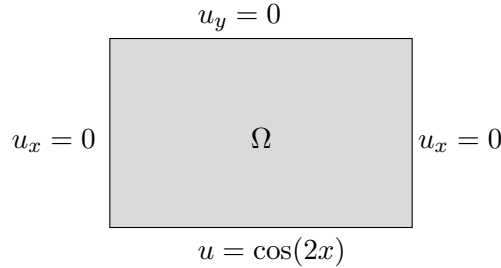


Figure 3: Two dimensional rectangle domain with homogeneous Neumann side boundaries, Example 1.

Consider the boundary value problem in a rectangular domain with homogeneous Neumann side boundaries as shown in Figure 3. This problem can be solved using separation of variables and solution is given as,

$$u(x, y) = \frac{\cosh(4\pi(y - b)/a)}{\cosh(4\pi b/a)} \cos(4\pi x/a), \quad (58)$$

To validate observer based approach this analytical solution given in (58) along with homogeneous Neumann boundary condition is used as Cauchy data on the top boundary Γ_T . Using this Cauchy data state observer algorithm is run for a number of iterations to recover the unknown boundary data on the bottom boundary Γ_B . Figure 4 shows the comparison of exact solution and the one recovered by using Cauchy data on Γ_B and observer algorithm.

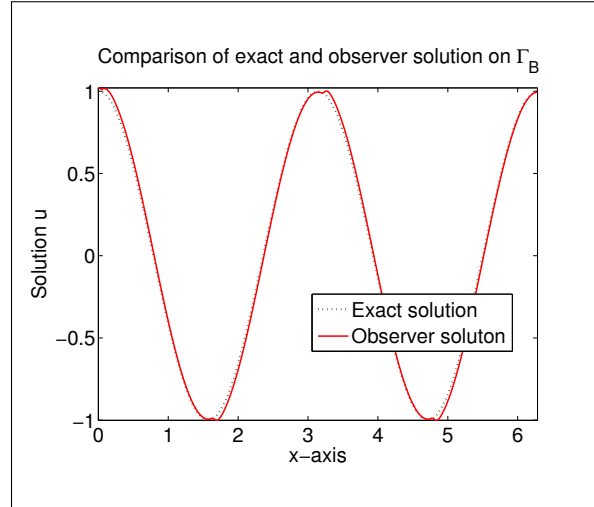


Figure 4: Comparison of exact and observer constructed solution on the bottom boundary Γ_B .

4.5.2 Example 2: Homogeneous Dirichlet side boundaries

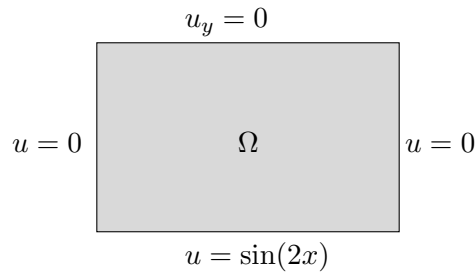


Figure 5: Two dimensional rectangle domain with homogeneous Dirichlet side boundaries, Example 2.

Consider the boundary value problem with homogeneous Dirichlet side boundaries as shown in Figure 5. Analytical solution is given as,

$$u(x, y) = \frac{\cosh(4\pi(y - b)/a)}{\cosh(4\pi b/a)} \sin(4\pi x/a), \quad (59)$$

Now using this analytical solution along with homogeneous Neumann boundary condition on the top boundary Γ_B , observer algorithm is run for a number of iterations to recover the unknown Dirichlet boundary data on Γ_B . Figure 6 shows the comparison of the exact and observer constructed solution on Γ_B .

4.5.3 Example 3: Linear combinations of example 1 and 2

It is easy to see that any linear combination of above two example problems can be solved using observer based technique. In other words any Dirichlet boundary data on Γ_B that can be represented as a trigonometric Fourier series can be recovered using observer based approach given homogeneous Dirichlet, Neumann or Robin kind of side boundaries. The requirement of such homogeneous side boundaries suggest that there are no active sources on the side boundaries which is indeed the case for many applications like electrocardiography (ECG) where objective is to find heart potential which is deep inside the body from the only available

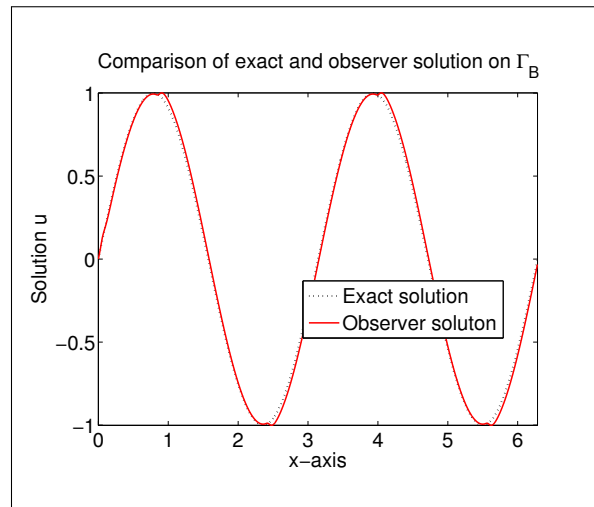


Figure 6: Comparison of exact and observer constructed solution on the bottom boundary Γ_B .

ECG data on a limited part of body torso [3, 4]. The observer based approach is the optimum technique in cases where there is no information available on the side boundaries. Figure 7 compares the exact solutions in different test cases to the one obtained by using observer. Numerical solution was achieved using homogeneous Neumann boundaries on Γ_L, Γ_R and Γ_T and non zero Dirichlet data on Γ_B . The observer solution was constructed using only the Cauchy data on Γ_T .

5 Conclusion

Cauchy problem for Laplace equation is a steady state problem. The design of a dynamical systems technique like observer for this problem is challenging and the idea to use one of the space variables as a time-like variable has not been considered before.

Different from standard approaches to tackle this problem, an iterative observer is constructed in infinite dimensional setting on a rectangle domain without introducing an extra time variable. Laplace equation is presented as a state equation with state operator matrix \mathcal{A} . Conditions for the existence of strongly continuous semigroup generated by \mathcal{A} are provided. Further the conditions for the existence of observer gain are provided in a proper functional analysis framework. Numerical results are provided for various example test cases. This paper reflects the possibility of considering a steady state problem from a dynamical theory perspective by using one of the space variables as a time-like variable.

Possible future work includes extension of the proposed method to three dimensions with more complicated domains using the interesting observability result obtained in this work.

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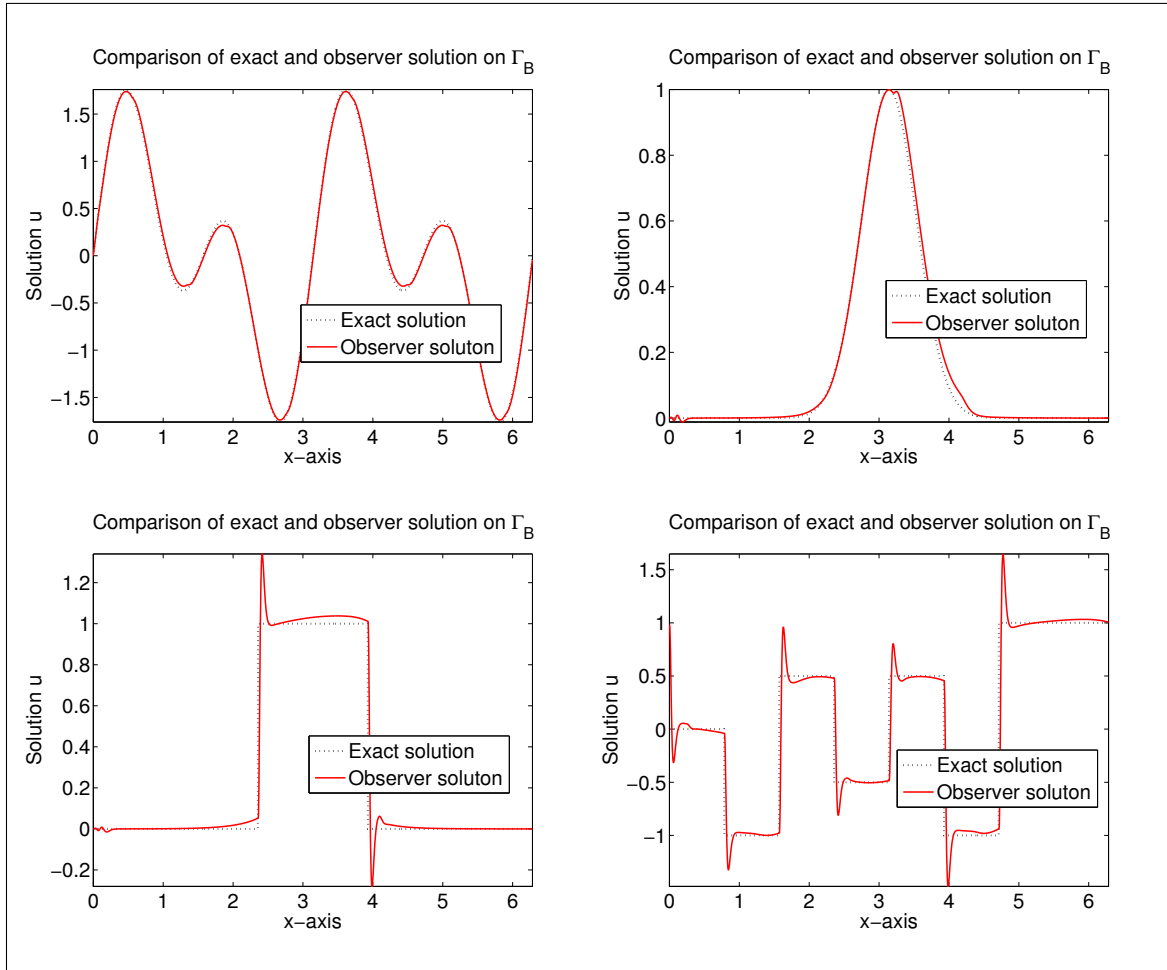


Figure 7: Comparison of exact and observer constructed solution on the bottom boundary Γ_B .

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